Very Short Introduction to Calibrated Geometry

Jakob R. Stein

UCL

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- Can often distinguish geometries by special sub-manifolds, e.g. geodesics, or foliations.
- May be interested in Riemannian manifolds (M, g) with distinguished forms, e.g. special holonomy.
- Calibrated geometry relates these forms to the volume form of the induced metric $\iota^* g$ of an immersed submanifold $\iota : N \hookrightarrow M$.

Definition (Calibration)

$\varphi \in \Omega^k(M)$ of Riemannian manifold (M, g) is a *calibration* if:

- $d\varphi = 0$
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We will see more interesting examples later, but the point is that calibrations are fairly abundant.

Definition (Calibrated sub-manifold)

Let φ be a calibration on Riemannian manifold (M, g). Submanifold $N \subset M$ *calibrated* w.r.t to φ if $\varphi(e_1, \dots, e_k) = 1$ for oriented orthonormal basis for TN (i.e. φ is the volume form).

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Example

In the previous example, on Euclidean $\mathbb{R}^n$ choose coordinates x_i so that k-form $\varphi = dx_1 \wedge \ldots \wedge dx_k$. Then φ calibrates the plane $\{x \in \mathbb{R}^n \mid x_{k+1} = \ldots = x_n = 0\}$.

Lemma (Application)

Calibrated sub-manifolds are minima of the volume functional over a fixed homology class.

For N to be a critical pt of volume functional is a 2nd order (elliptic) p.d.e. but calibrated condition 1st order p.d.e.

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Example (Kähler i.e. $\mathrm{Hol} = U(n)$)

Let (M, g, ω) be Kähler: then $\frac{1}{k!}\omega^k$ is a calibration and its calibrated submanifolds are the complex k-dimensional submanifolds.

Example (Calabi-Yau i.e. $\mathrm{Hol} = SU(n)$)

Let (M, g, ω, Ω) be Calabi-Yau. Then $\mathrm{Re}\Omega$ is a calibration, and calibrated submanifolds are *special Lagrangian* (SL) i.e. N is SL iff $\omega|_N \equiv \mathrm{Im}\Omega|_N \equiv 0$.

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Theorem (McLean)

Let (M, g, ω, Ω) be Calabi-Yau, and N compact special Lagrangian submanifold. Then the moduli space $\mathcal{M}(N)$ of special Lagrangian deformations of N is a smooth manifold of dimension $b_1(N)$.

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Example ($\mathrm{Hol} = G_2$)

Let (M, g, ϕ) be G_2. Then $\phi, *\phi$ are calibrations. Calibrated submanifolds are called *associatives* and *co-associatives* respectively. N is co-associative iff $\phi|_N \equiv 0$.

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Example ($\mathrm{Hol} = Spin(7)$)

Let (M, g, Φ) be $Spin(7)$. Then Φ is a calibration, and calibrated submanifolds are called Cayley submanifolds.

Analytic Problems

- Compactness of the space of minimal/calibrated sub-manifolds (e.g. integral currents)
- Immersions vs. embedding
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- Constructions of calibrated submanifolds (e.g. with symmetries, or by gluing).
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Applied mathematics

- Counting invariants for special holonomy (see conjectures of Joyce, Donaldson/Segal).
- Instantons/monopoles: bubbling along calibrated sets (see paper of Tian, work of Haydys).
- Problems in minimal sub-manifolds/mean curvature flow (e.g. Hsiang/Lawson conjecture).