Introduction to Gauge Theory and Invariants

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Plan

- History
- Mathematical background
- Motivation
- Example: Casson invariant

History

Physics

Equations of motion with *local symmetries*, i.e. some differential equation $F(x) = 0$ invariant under Lie group G of transformations, where the element $g \in G$ acting depends on x.

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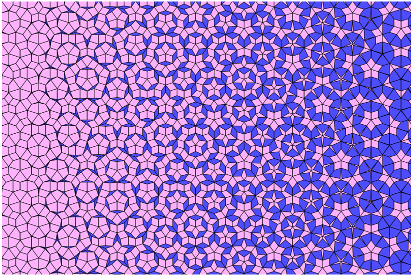

Local symmetries

Background: Topology

Principal bundles

Let G be a Lie group. A principal G-bundle P over manifold M is a fibre bundle with fibre G, and a (free, transitive) action of G on the fibre: i.e. Locally (on open sets $U \subset M$) we have $P|_U \cong U \times G$, with some transition maps $G \to G$ between the fibres over different open sets.

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The gauge group $\mathcal{G} := Aut(P)$, i.e. the group of G-equivariant maps $P \to P$ that preserve the fibres of P. Since it preserves the fibre, and each fibre is generated by the action of G, an element in $\mathcal{G}$ can also be thought of as equivariant map from P to G.

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Example

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Background: Differential Geometry

Connections $A \in \mathcal{A}$

A **connection** (locally on U) at $(x, g) \in U \times G$ is given by $A = g^{-1}dg + gA_U g_{-1}$ where A_U is a one-form on U with values in the lie algebra $\mathfrak{g}$ of G. Explicitly, if $\mathrm{x} = (x^1, \ldots, x^n)$ are coordinates on U, and $E_i : U \to \mathfrak{g}$, $A_U|_{\mathrm{x}} = \sum_i E_i(\mathrm{x}) \otimes dx^i$. Warning: this depends on our choice of coordinates. We will denote the space of connections on a fixed bundle $\mathcal{A}$.

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$\phi \in \mathcal{G}$ acts via the adjoint representation on F_A, i.e. $F_{\phi(A)} = \phi^{-1} F_A \phi$.

Background: Differential Geometry (cont.)

Flat connections

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Fact

$\mathcal{A}$ is an **affine** space w.r.t to $\mathrm{Ad}P \otimes \Omega^1(M)$: i.e.
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Theorem

There is a bijection of sets $\{A \in \mathcal{A} \mid F_A = 0\}/\mathcal{G} = \hom(\pi_1(M), G)/G$, *where* G *acts via conjugation on* $\hom(\pi_1(M), G)/G$.

Motivations

In finding invariants, we usually take some bundle our manifold with $G = U(1), SU(2)$ or $SO(3)$. The $U(1)$ case, while interesting, is somewhat different to the $SU(2)/SO(3)$ cases. Examples of the latter:

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Now we are armed with some facts about connections, we will look at an example: a simple type of gauge invariant called the **Casson invariant** (interpretation due to Taubes).

Casson Invariant

Let M be a compact 3-manifold with a trivial bundle $P = M \times SU(2)$. Then $\mathcal{A} - \Gamma = \mathfrak{su}(2) \otimes \Omega^1(M)$. Let $\mathcal{A}^\# \subset \mathcal{A}$ denote the subset where $\mathcal{G}$ acts with stabiliser ± 1 and denote $\mathcal{B} := \mathcal{A}/\mathcal{G}$ (likewise $\mathcal{B}^\#$) We will assume M is a homology-sphere, i.e. $H_*(M) = H_*(S^3)$.

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Casson Invariant $C_a(M)$

Informally, the Casson invariant is "the signed count of flat connections on P", or "the Euler characteristic of $\mathcal{B}^\#$".

Analogy

If (M, g) Riemannian manifold, v (non-degenerate) v.field, then:

$$\chi(M) = \sum_{p | v(p)=0} \text{sign}(\det \nabla v|_p)$$

Here ∇ is the Levi-Cevita connection on (M, g). At each point p, $\nabla v|_p$ is a finite dimensional matrix on $T_p M$.

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Strategy

Define a one-form on $\mathcal{B}^{\#}$, use some metric to make it a vector field, and find a way to take its derivative, and make sense of sign of determinants in infinite dimensional space $T_{[A]}\mathcal{B}^{\#}$.

Identify $T_A\mathcal{A}$ with the vector space $\mathfrak{su}(2) \otimes \Omega^1_p(M)$. We use the one-form:

$$\mathfrak{f} : \mathcal{A} \to T^*\mathcal{A}$$

$$\mathfrak{f}_A(a) := \int_M \operatorname{tr}(a \wedge F_A)$$

This descends to $\mathcal{B}$, and the flat connections are its zeroes.

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There is a natural notion of a covariant derivative (omitting details) to get $\nabla \mathfrak{f}_A$, an infinite-dimensional matrix on $T_{[A]}\mathcal{B}^{\#}$.

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$\nabla \mathfrak{f}_A$ *is closed, has real, discrete eigenvalues, and the set of eigenvalues has no accumulation points.*

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Spectral Flow

So for such $[A], [A'] \in \mathcal{B}^{\#}$, pick a path between them in $\mathcal{B}^{\#}$, define:

$$n(A, A') = (\#\text{ eigenvalues of } \nabla \mathfrak{f}_A \text{ crossing 0 from the right} - \\ \#\text{ eigenvalues of } \nabla \mathfrak{f}_A \text{ crossing 0 from the left}) \bmod 2 \quad = \pm 1$$

One can show this is well-defined and path-independent.

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One can show this is well-defined and path-independent. Pick some non-trivial flat connection $[A] \in \mathcal{B}^{\#}$, then define the (unsigned) Casson invariant:

$$C_a(M) = \sum_{[A'] \neq [A] | F_{A'} = 0} n(A, A')$$

It turns out this is independent of the choices of Riemannian metric, and the flat connection used.

Closing Remarks

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- Casson originally defined his invariants without gauge theory, using something called a Heegaard splitting (pictured).
- This invariant is just the beginning of a wider story of gauge theory, but this is the model example.